The Critical Density of Friedmann's Equation is the Density of the Universe at its Schwarzschild Radius

In a recent conversation regarding the age and development of the universe, it was asserted that the critical density term (ρ_c) as used in Friedmann's equation changes with time. This appears to be a commonly accepted concept, and indeed, an internet search for the phrase "critical density of the universe" will return several thousand pages of assurances that such variation of the critical density with time is in fact the case. The only explanation provided for this variance is that ρ_c can be written opposite the time dependent Hubble constant in the Friedmann equation when k equals zero as expressed in equation (1).

$$\rho_c = \frac{3H^2}{8\pi G} \quad (1)$$

The circular reasoning inherent in this explanation should be glaringly obvious, and yet I have searched in vain for a single mention of it in any publication – scholarly or popular. All we need to do to reveal the circle is ask two very simple questions.

First, we must ask why the term ρ_c is used in equation (1) instead of ρ. The derivation of equation (1) usually begins by writing the Friedmann equation with a zero cosmological constant as in equation (2).

$$H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} \quad (2)$$

A zero value is then substituted for k to produce equation (3).

$$\rho = \frac{3H^2}{8\pi G} \quad (3)$$

ρ_c is then defined as the density of the universe when k equals zero and substituted in place of ρ to produce equation (1).

Having received an answer for our first question, we could then ask why k should be set to equal zero. The answer given would be that a zero value for k describes a universe that is flat or, to be more specific, a universe in which the actual density is the same as the critical density.

The circle is now revealed. The critical density, in this case, is defined as the density when k equals zero while a zero value is only attributed to k if the universe is at its critical density. We could spend many maddening hours repeating these questions and never receive different answers. What is the critical density? The density of the universe when k equals zero. When does k equal zero? When the universe is at its critical density. Etc. etc. etc. The circle just goes on and on.

To break the circle and return to sound logic, we must discover an alternative definition of the critical density. It is my contention that the critical density should be defined as that density which the universe would display if it occupied a sphere described by its own Schwarzschild radius. In other words, the critical density of the universe is identical to its Schwarzschild density (ρ_s). This definition follows directly from three observations. First, that equation (1) can be resolved to the equation for the Schwarzschild radius. Second, that the calculated estimates of the critical density are equivalent to the calculated estimates of the Schwarzschild density. And third, that the correlation between the Schwarzschild density and the critical density has been noted from time to time within the scientific literature.

For the first observation, we will begin by rewriting equation (1).

$$\rho_c = \frac{3H^2}{8\pi G} \quad (1)$$

We can then add equation (4) to our consideration which describes the Hubble constant in terms of the speed of light and the radius of the observable universe.

$$H = \frac{c}{R} \quad (4)$$

Substituting (4) into (1) produces:

$$\rho_c = \frac{3c^2}{8\pi G R^2} \quad (5)$$

At this point, we can consider that the density of a sphere is described by:

$$\frac{3M}{4\pi R^3} \quad (6)$$

A sphere having the same density as the critical density of the universe can therefore be described as:

$$\frac{3M}{4\pi R^3} = \frac{3c^2}{8\pi G R^2} \quad (7)$$

Equation (7) can be simplified and solved for R to produce:

$$R = \frac{2GM}{c^2} \quad (8)$$

This, as should be recognized, is the exact formula for the Schwarzschild radius, and we can thereby see that a universe having a density corresponding to that of the critical density would also have a radius corresponding to that of its Schwarzschild radius. It thus follows that the critical density of the universe is identical to its Schwarzschild density.

The accuracy of this first observation can be validated by the second in which actual calculations of the two densities are compared for equivalence. The equation for determining the Schwarzschild density can be derived from equation (8) as:

$$\rho_s = \frac{3c^6}{32\pi G^3 M^2} \quad (9)$$

By attributing an estimated mass to the universe of 1×10^{53} kg, we can solve equation (9) to discover that the Schwarzschild density of the universe is 7×10^{-30} g/cm^3. This value falls directly within the range of estimates that Alan Guth presented for the critical density in his book, *The Inflationary Universe*. The equivalence between the calculations of the critical density and the Schwarzschild density cannot be denied, and we find that our first observation is confirmed by the second.

The third observation lends even greater credence to my contention. The correlation between the critical density and the Schwarzschild density is rarely mentioned in the scientific literature, and it is even more rare to find it referenced directly. Such references are present, however, and careful study will reveal that that the scientific community fully confirms the conclusion that the critical density of the universe is in fact the same as its Schwarzschild density.

For example:

"All the matter in the universe must ultimately collapse to a single black hole ... The general theory of relativity says that nothing can stop the collapse, because after the Schwarzschild radius is reached, the direction in time from the past to the future is necessarily tied to the continuing collapse down to a point."
- Lloyd Motz, *The End of the Universe and the End of Time*

"Gravitational collapse confronts physics with its greatest crisis ever. At issue is the fate, not of matter alone but of the universe itself ... Collapse, moreover, is not unique to the large scale dynamics of the universe. A white dwarf star or a neutron star of more than critical mass is predicted to undergo gravitational collapse to a black hole. Sufficiently many stars falling sufficiently close together at the center of the nucleus of a galaxy are expected to collapse to a black hole ... The process that makes a black hole is predicted to provide an experimental model for the gravitational collapse of the universe itself."
- John A. Wheeler, *Beyond the End of Time*

"We now come to the fourth problem, the critical density of matter ... If the actual density today were higher than that value, the expansion of the universe would end after a certain future time, because the attraction among matter would then be strong enough to turn the expansion into a contraction ... Conversely, if the actual density today were below the critical value, then the gravitational attraction would not be strong enough to turn the expansion around."
- Victor F. Weisskopf, *The Origin of the Universe*

"The universe is a system of strong self-gravity. The geometry of the universe is determined by self-gravity, and the size of the universe is at least its gravitational radius."
- R. Brustein and G. Veneziano, "A Causal Entropy Bound for a Spacelike Region"

"Thus, the relativistic critical-density condition for the expanding universe of mass M„ corresponds to the radius being equal to the Schwarzschild radius of a static universe of mass M„."
- Y.A. Yoler, *Perception of Natural Events by Human Observers*

It is apparent from these three observations that the critical density of the universe and the Schwarzschild density of the universe are one and the same. From this conclusion, it follows that the critical density does not fluctuate with time as has been claimed. The Schwarzschild density depends solely upon an object's mass and remains constant for a given mass regardless of its size or time. The same must be true of the critical density as well. The fact that ρ_c can be written opposite the time dependant Hubble constant in the Friedmann equation no more causes the critical density to vary with time than solving the same equation for π would cause the area of the circle to vary with time.